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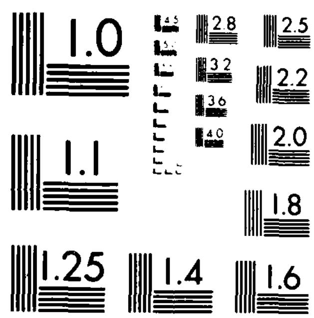

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DIFFUSION ON VISCOUS FLUIDS, EXISTENCE AND ASYMPTOTIC PROPERTIES OF SOLUTIONS

H. Beirão-da-Veiga

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

~~We~~ considers the motion of a mixture of two fluids, with a diffusion effect obeying Fick's law, for the derivation of the model (1.1) see Section 1 and references [2], [4], [5] and [6]. ~~We~~ consider, the full non-linear problem (i.e., we don't omit the term λ^2 term in equation (1.1_1). Moreover, we ~~don't~~ assume that ~~λ/μ~~ lambda/micron is small. ~~We~~ He prove, the existence of a (unique) local solution, the existence of a global solution for small data, and the exponential decay to the equilibrium solution; see Theorem A, Section 1.

AMS(MOS) Subject Classifications: 35K55, 76D99, 76R99.

Key Words: Non-homogeneous viscous fluids, non-linear parabolic equations, initial-boundary value problems

Work Unit Number 1 (Applied Analysis)

*Department of Mathematics, University of Trento, 38050 Povo (Trento) Italy.

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SIGNIFICANCE AND EXPLANATION

In this paper we consider the motion of a continuous medium consisting of two components, say water and a dissolved salt, with a diffusion effect obeying Fick's law. We denote by v,w,ρ,π,μ,λ the mean-volume velocity, mean-mass velocity, density, pressure, viscosity and diffusion constant, respectively. By using Fick's law we eliminate w from the equations and we obtain (1.1), where p is the modified pressure; see Section 1 and references [2], [4],[5] and [6]. The initial boundary conditions are given by equation (1.2).

Kazhikhov and Smagulov [5],[6] consider equation (1.1) for a small diffusion coefficient λ. More precisely, they assume that condition (1.3) holds; moreover, they omit the λ^2 term in equation $(1.1)_1$. Under these conditions they prove the existence of a unique local solution for the 3-dimensional motion (in the bi-dimensional case, solutions are global).

In our paper we consider the full equation (1.1), without assumption (1.3), and we prove: (i) the existence of a (unique) local solution; (ii) the existence of a global solution in time for small initial velocities and external forces, and for initial densities near-constant; (iii) the exponential decay (when $t \to +\infty$) of the solution (ρ,v) to the equilibrium solution $(\hat{\rho},0)$, if $f \equiv 0$. See Theorem A, Section 1.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DIFFUSION ON VISCOUS FLUIDS, EXISTENCE
AND ASYMPTOTIC PROPERTIES OF SOLUTIONS

H. Beirão-da-Veiga*

Main Notation

Ω : an open bounded set in R^3, locally situated on one side of its boundary Γ, a regular (say C^4) manifold.

$n = n(x)$: unit outward normal to Γ.

D_i, D_{ij}, D_t : $\partial/\partial x_i$, $\partial^2/\partial x_i \partial x_j$, $\partial/\partial t$.

$\| \;\|, (\;,\;)$: norm and scalar product in $L^2(\Omega)$.

H^k : Sobolev space $H^{k,2}(\Omega)$ with norm

$$\|\sigma\|_k^2 \equiv \sum_{l=0}^{k} \|D^l \sigma\|^2,$$

where

$$\|D^l\sigma\|^2 \equiv \sum_{|\alpha|=1} \|D^\alpha \sigma\|^2,$$

Further,

$$\|D^l\sigma\|_m^2 \equiv \sum_{|\alpha|=1} \|D^\alpha \sigma\|_m^2 .$$

H_0^1 : Closure of $C_o^\infty(\Omega)$ in $H^1(\Omega)$.

$\| \;\|_\infty$: norm in $L^\infty(\Omega)$.

L^2, H^k, H_0^1 : Hilbert spaces of vectors $v = (v_1, v_2, v_3)$ such that $v_i \in L^2, v_i \in H^k, v_i \in H_o^1$ $(i=1,2,3)$, respectively. Corresponding notation is used for other spaces of vector fields. Norms are defined in the natural way, and denoted by the symbols used for the scalar fields.

Let us introduce the following functional spaces (see, for instance, [7], [8] and [12] for their properties) =

*Department of Mathematics, University of Trento, 38050 Povo (Trento) Italy.

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$$H_N^k \equiv \{\sigma \in H^k : \frac{\partial \sigma}{\partial n} = 0 \text{ on } \Gamma \text{ and } \int_\Omega \sigma(x)dx = 0\}, \ k \geq 2 .$$

$$\mathcal{V} \equiv \{ v \in C_o^\infty(\Omega) : \text{div } v = 0 \text{ in } \Omega\},$$

$$\mathbf{H} = \{ v \in \mathbf{L}^2 : \text{div } v = 0 \text{ in } \Omega, \ v \cdot n = 0 \text{ on } \Gamma\},$$

$$\mathbf{V} = \{ v \in \mathbf{H}_0^1 : \text{div } v = 0 \text{ on } \Omega\}.$$

$\mathbf{H}$ and $\mathbf{V}$ are the closure of $\mathcal{V}$ in $L^2(\Omega)$ and $\mathbf{H}_o^1$, respectively. Moreover $\mathbf{L}^2 = \mathbf{H} + G$, where $G \equiv \{ \nabla p : p \in H^1(\Omega)\}$. Denoting by P the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{H}$, we define the operator $A \equiv -P\Delta$ on $D(A) \equiv \mathbf{H}^2 \cap \mathbf{V}$. One has $(Au,v) = ((u,v)) \equiv \sum_{i,j} (D_i U_j, D_i v_j)$, $\forall\, u \in D(A)$, $v \in V$.

The norms $\|\sigma\|_2$, $\|\Delta\sigma\|$ are equivalent in H_N^2, $\|\sigma\|_3$, $\|\nabla\Delta\sigma\|$ are equivalent in H_N^3 and $\|v\|_2$, $\|Av\|$ are equivalent in $D(A)$. We define $\|v\|_V^2 \equiv ((v,v))$; the norms $\|v\|_V$, $\|v\|_1$ are equivalent in V.

$L^2(0,T;X)$: Banach space of strongly measurable functions defined in $]0,T[$ with values in (a Banach space) X, for which

$$\|z\|^2_{L^2(0,T;X)} \equiv \int_0^T \|z(t)\|_X^2 \, dt < +\infty.$$

$C(0,T;X)$: Banach space of X-vector valued continuous functions on $[0,T]$ endowed with the usual norm $\|z\|_{C(0,T;X)}$.

μ : viscosity (a positive constant).

λ : diffusion coefficient (a positive constant).

$v(t,x)$, $v_o(x)$: mean-volume velocity. Initial mean-volume velocity.

$\rho(t,x)$, $\rho_o(x)$: density of the mixture. Initial density.

Further,

$$m \equiv \inf_{x\in\Omega} \rho_o(x), \quad M \equiv \sup_{x\in\Omega} \rho_o(x),$$

$$\hat{\rho} \equiv \frac{1}{|\Omega|} \int_\Omega \rho_o(x)dx.$$

We assume that $m > 0$.

$\pi(t,x)$, $p(t,x)$: pressure. Modified pressure

$$p = \pi + \lambda\, v \cdot \nabla\rho - \lambda^2 \Delta\rho + \lambda(2\mu+\mu')\Delta \log \rho.$$

$f(t,x)$: external mass-force.

We denote by c, $\bar{c}$, c_0, c_1, c_2, ... positive constant, depending at most on Ω and on the parameters μ, λ, m, M and $\hat{\rho}$. It is easy to derive at any stage of the proofs, the explicit dependence of the constants on the parameters.

For convenience we sometimes denote different constants by the same symbol c. Otherwise, we utilize the symbols $\bar{c}$, c_k, $k \in \mathbb{N}$.

1. Main results. In this paper we consider the motion of a viscous fluid consisting of two components, say, saturated salt water and water. The equations of the model are obtained, for example, in [2], [4], [5], and [6]. Let us give a brief sketch. Let ρ_1, ρ_2 be the characteristic densities (constants) of the two components, $v^{(1)}(t,x)$ and $v^{(2)}(t,x)$ *their velocities, and* $e(t,x)$, $d(t,x)$ the mass and volume concentration of the first fluid. We define the density $\rho(t,x) \equiv d\rho_1 + (1-d)\rho_2$, and the mean-volume and mean-mass velocities $v \equiv d\, v^{(1)} + (1-d)\, v^{(2)}$, $w \equiv e\, v^{(1)} + (1-e) v^{(2)}$. Then the equations of motion are

$$\begin{cases} \rho[D_t w + (w\cdot\nabla)w - f] - \mu\Delta w - (\mu+\mu')\nabla \operatorname{div} w = -\nabla\pi, \\ \operatorname{div} v = 0 , \\ D_t\rho + \operatorname{div}(\rho w) = 0 . \end{cases}$$

On the other hand, Fick's diffusion law (see [2]) gives $w = v - \lambda\rho^{-1}\nabla\rho$. By elimination of w in the preceeding equations one gets, after some calculations,

$$(1.1)\qquad \begin{cases} \rho(D_t v + (v\cdot\nabla)v) - \mu\Delta v - \lambda\, [(v\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)v] + \\ \quad + \dfrac{\lambda^2}{\rho}\, [(\nabla\rho\cdot\nabla)\nabla\rho - \dfrac{1}{\rho}\, (\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho] = -\nabla p + \rho f, \\ D_t\rho + v\nabla\rho - \lambda\Delta\rho = 0 , \\ \operatorname{div} v = 0 . \end{cases}$$

We want to solve system (1.1) in $Q_T \equiv \,]0,T[\, \times \Omega$. Here p is the modified pressure. We add to system (1.1) the following initial boundary-value conditions.

$$
(1.2)\qquad \begin{cases} v = 0 & \text{on }]0,T[\times \Gamma, \\ \dfrac{\partial\rho}{\partial n} = 0 & \text{on }]0,T[\times \Gamma, \\ v_{|t=0} = v_o(x) & \text{in } \Omega, \\ \rho_{|t=0} = \rho_o(x) & \text{in } \Omega . \end{cases}
$$

The first two conditions mean that there is no flux through the boundary.

In [5], [6] Kazhikhov and Smagulov consider the simplified system obtained from (1.1) by omitting the term containing λ^2; moreover, they assume that

$$
(1.3)\qquad \lambda < \frac{2\mu}{M-m} .
$$

Under these conditions Kazhikhov and Smagulov state the existence of a local solution in time (global in the bidimensional case).

In our paper we take into account the full equation (1.1), and omit the condition (1.3). For this more general case we prove: (i) the existence of a (unique) local solution for arbitrary initial data and external face field; (ii) the existence of a unique global strong solution, for small initial data and external force field. Moreover, if $f \equiv 0$, the solution (ρ,v) decays exponentially to the equilibrium solution $(\hat{\rho},0)$. More precisely we prove the following result:

<u>Theorem A. Let</u> $v_o \in \mathbf{V}, \rho_o - \hat{\rho} \in H^2_N$, $f \in L^2(0,T;\, L^2)$. <u>Then there exists</u> $T_1 \in\,]0,T]$ <u>such that problem</u> (1.1), (1.2) <u>is uniquely solveable in</u> Q_{T_1}. <u>Moreover</u> $v \in L^2(0,T_1;\, \mathbf{H}^2) \cap C\,(0,T_1;\, \mathbf{V})$, $D_t v \in L^2\,(0,T_1;\, \mathbf{H})$, $\rho-\hat{\rho} \in L^2(0,T_1; H^3_N) \cap C\,(0,T_1;\, H^2_N)$, $D_t\rho \in L^2(0,T_1;\, H^1)$ <u>and</u> $m \le \rho(t,x) \le M$.

<u>Moreover, there exist positive constants</u> k_1, k_2, and k_3 <u>depending at most on</u> Ω, μ, λ <u>and on the mean density</u> $\hat{\rho}$ [(1)] <u>such that that if</u>

$$
(1.4)\qquad \|v_o\|_1 + \|\rho_o-\hat{\rho}\|_2 \le k_1,
$$

(1) Or, equivalently, depending on the total amount of mass $|\Omega|\,\hat{\rho} = \int_\Omega \rho_o(x)dx$.

and

(1.5) $$\|f\|_{L^\infty(0,+\infty;L^2)} \le k_2,$$

<u>then the solution is global in time</u>. <u>If</u> $f \equiv 0$, <u>the solution</u> (ρ,v) <u>decays exponentially to the equilibrium solution</u> $(\hat{\rho},0)$, <u>i.e.</u>

(1.6) $$\|v(t)\|_1 + \|\rho(t)-\hat{\rho}\|_2 \le (\|v_o\|_1+\|\rho_o-\hat{\rho}\|_2)\, e^{-k_3 t},$$

<u>for every</u> $t \ge 0$.

Theorem A also holds for coefficients μ,λ regularly dependent on ρ, v, provided they are strictly positive and bounded, in a neighborhood of the range of values of the initial data $\rho_o(x)$, $v_o(x)$. This generalization can be done without any difficulty. Moreover, with standard techniques, one can prove that the solutions have more regularity (up to C^∞) if the data are sufficiently regular and the usual compatibility conditions hold.

Local existence in the general case (i.e. with the λ^2term and without (1.3)) was proved in the <u>inviscid</u> case by Beirão-da-Veiga, Serapioni and Valli in [1]. a similar result, in the viscous case and for $\Omega = \mathbb{R}^3$, was proved by Secchi [11]. For another approach (concerning Greffi's model) see [10].

2. <u>The linearized equations</u>. *We start by proving the following theorem:* <u>Theorem 2.1</u> <u>Let</u> $\rho(t,x)$ <u>be a measurable function verifying</u>

(2.1) $$0 < m \le \rho(t,x) \le M \quad , \quad \text{a.e., in } Q_T, \text{ let } F \in L^2(0,T;H) \text{ and } v_o \in V.$$

<u>Then there exists a (unique) strong solution</u> v <u>of problem</u>

(2.2) $$\begin{cases} \rho\, D_t v - \mu\Delta v = -\nabla p + F & \text{in } Q_T, \\ \operatorname{div} v = 0 & \text{in } Q_T, \\ v = 0 & \text{on }]0,T[\times\Gamma, \\ v_{|t=0} = v_o(x) & \text{in } \Omega. \end{cases}$$

<u>Moreover</u> $v \in L^2(0,T;D(A)) \cap C(0,T;V)$, $D_t v \in L^2(0,T;H)$ <u>and</u>

(2.3) $$\mu\|v\|^2_{C(0,T;V)} + m\,\|v'\|^2_{L^2(0,T;H)} + \frac{m\mu^2}{4M^2}\|Av\|^2_{L^2(0,T;H)} \le \mu\|v_o\|^2_V + \left(\frac{2}{m} + \frac{m}{2M^2}\right)\|F\|^2_{L^2(0,T;H)}$$

Proof. Let us write equation (2.2) in the equivalent form(2.4)

$P(\rho D_t v) + \mu A v = F \ , \ v_{|t=0} = v_o(X)$.

For brevity let us put

$$X = \{v : v \in L^2(0,T;D(A)),\ v' \in L^2(0,T;H)\} \ .$$

From well known results (see [9], Vol I, Chapter I : Theorem 3.1 with $y = H$, $X = D(A)$, $j = 0$; and (2.42) Proposition 2.1) it follows that $X \hookrightarrow C(0,T;V)$.

We start by proving the a priori bound (2.3); an essential device is to introduce a parameter ε_o in order to conveniently balance the estimates. In H take the inner product of (2.4) with $D_t v + \varepsilon_o A v$, $\varepsilon_o > 0$. Since $(v', Av) = 2^{-1} D_t \|v\|_V^2$ one gets

$$(2.5) \qquad m \|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \varepsilon_o \mu \|Av\|^2 \le$$

$$\le \|F\|\,\|D_t v\| + \varepsilon_o \|F\|\,\|Av\| + \varepsilon_o M \|D_t v\|\,\|Av\| \ .$$

By using the inequalities $\|F\|\,\|D_t v\| \le 4^{-1} m \|D_t v\|^2 + m^{-1}\|F\|^2$, $\|F\|\,\|Av\| \le 4^{-1}\mu \|Av\|^2 + \mu^{-1}\|F\|^2$ and $\|D_t v\|\|Av\| \le (4M)^{-1}\mu \|Av\|^2 + \mu^{-1} M \|D_t v\|^2$ one gets

$$(2.6) \qquad \frac{3}{4} m \|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \varepsilon_o \frac{\mu}{2}\|Av\|^2 \le$$

$$\le \left(\frac{1}{m} + \frac{\varepsilon_o}{\mu}\right)\|F\|^2 + \frac{\varepsilon_o M^2}{\mu}\|D_t v\|^2 \ .$$

Now fix $\varepsilon_o = (4M^2)^{-1} m\mu$ and integrate equation (2.6) on $(0,T)$. This gives the a priori bound (2.3).

Define $\|v\|_X^2 \equiv$ "left hand side of equation (2.3)", $Y \equiv L^2(0,T;H) \times V$, and $\|(F,v_o)\|_Y^2 \equiv$ "right hand side of (2.3)".

We solve (2.4) by the continuity method. Define $\rho\alpha \equiv (1-\alpha)\hat{\rho} + \alpha\rho$, $\alpha \in [0,1]$. Clearly ρ_α verifies condition (2.1), for any α. Define $T_\alpha \equiv (1-\alpha)\bar{T} + \alpha T$, where

$$Tv \equiv (P(\rho D_t v) - \mu A v,\ v_{|t=0}) \in Y,$$
$$\bar{T}v \equiv (P(\bar{\rho} D_t v) - \mu A v,\ v_{|t=0}) \in Y.$$

Finally denote by γ the set of values $\alpha \in [0,1]$ for which problem (2.2) is solvable in X for every pair $(F,v_o) \in Y$. Clearly $o \in \gamma$, because for this value of the parameter equation (2.6) because the linearized Navier-Stokes equation. Let us verify that γ is open and closed.

<u>γ is open</u>. Let $\alpha_o \in \gamma$ and denote by $G(F,v_o) \equiv v$ the solution v of problem $T_{\alpha_o} v = (F,v_o)$. From (2.3) one gets $G \in L(Y;X)$ ([2]) with $\|G\|_{Y,X} \leq 1$. Equation $T_{\alpha_o+\varepsilon}\, v = (F,v_o)$ can be written in the form

$$[1-\varepsilon\, G(\bar{T}-T)]v = G(F,v_o). \tag{2.7}$$

Since $\|G(\bar{T}-T)\|_{X,Y} \leq \|\bar{T}-T\|_{X,Y}$, equation (2.7) is solvable for $|\varepsilon| < \|\bar{T}-T\|^{-1}_{X,Y}$ (by a Neuman expansion).

<u>γ is closed</u>. Let $\alpha_m \in \gamma$, $\alpha_m \to \alpha_o$, and let V_n be the solution of $T_{\alpha_n} v_n = (F,v_o)$. From (2.3) one has $\|v_n\|_X \leq \|(F,v_o)\|_Y$. Since X is an Hilbert space there exists a subsequence $v_\nu \to v \in X$, weakly in X. From T, $\bar{T} \in L(X;Y)$ one has $\bar{T}v_\nu \to \bar{T}v$, $Tv_\nu \to Tv$ weakly in Y. Hence $T_{\alpha_\nu} v_\nu \to T_{\alpha_o} v$, i.e. $T_{\alpha_o} v = (F,v_o)$.

Let us now return to problem (1.1). Define

$$F(\rho,v) \equiv P\,\{-\rho(v\cdot\nabla)v + \lambda\,[(v\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)v] + \frac{\lambda^2}{\rho}\,[(\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho] + \rho f\}\,. \tag{2.8}$$

For convenience we will use in the sequel the translation

$$\rho = \hat{\rho} + \sigma. \tag{2.9}$$

Recall that $\hat{\rho}$ is a given constant. To solve problem (1.1) and (1.2) in our functional framework is equivalent to finding $v \in L^2(0,T;D(A))$, $v' \in L^2(0,T;H)$ and $\sigma \in L^2(0,T;H^3_N)$, $\sigma' \in L^2(0,T;H^1)$ such that

$$\begin{cases} P(\rho D_t v) - \mu A v = F(\rho,v), \\ v_{|t=0} = v_o(x), \\ D_t\sigma - \lambda\nabla\sigma = -v\cdot\nabla\sigma, \\ \sigma_{|t=0} = \sigma_o(x), \end{cases} \tag{2.10}$$

([2]) Banach space of linear continuous operator from Y into X, with norm $\|\ \|_{Y,X}$.

where $v_o \in V$ and $\sigma_o(x) - \hat{\rho} \in H^2_N(\Omega)$, are given. Note that from the above conditions on σ it follows that $\sigma \in C(0,T; H^2_N)$.

We solve (2.10) by considering the linearized problem

$$
(2.11) \qquad \begin{cases} P(\rho D_t v) - \mu\Delta v = F(\bar{\rho},\bar{v}) \equiv \bar{F}, \\ v_{|t=0} = v_o(x), \\ D_t\sigma - \lambda\Delta\sigma = -\bar{v}\cdot\nabla\bar{\sigma}, \\ \sigma_{|t=0} = \sigma_o(x), \end{cases}
$$

and by proving the existence of a fixed point $(\bar{\rho},\bar{v}) = (\rho,v)$ for the map $(\bar{\rho},\bar{v}) \to (\rho,v)$ defined by (2.11).

In order to get a sufficiently strong estimate for the linearized equation $(2.11)_3$ we take in account the particular form of the date $\bar{v}\cdot\nabla\bar{\sigma}$. As for estimate (2.3) we will introduce a balance parameter $\varepsilon > 0$.

<u>Theorem 2.2</u> <u>Assume that</u> $\bar{v} \in L^2(0,T;H^2) \cap C(0,T;H^1_0)$ <u>and that</u> $\bar{\sigma} \in L^2(0,T;H^3_N) \cap C(0,T;H^2_N)$. <u>Then the solution</u> $\sigma \in L^2(0,T;H^3_N)$, $\sigma' \in L^2(0,T;H^1)$ <u>of problem</u> $(2.11)_3$, $(2.11)_4$ <u>verifies the estimate</u>

$$
(2.12) \qquad \|\sigma\|^2_{C(0,T;\,H^2_N)} + \|\sigma\|^2_{L^2(0,T;H^3_N)} \le
$$

$$
\le c_2\|\sigma_o\|^2_2 + c_3\,\varepsilon^{-3}T[\|\bar{v}\|^6_{C(0,t;H^2)} + \|\nabla\bar{\sigma}\|^6_{C(0,T;H^1)}]
$$

$$
+ c_3\,\varepsilon\,[\|\bar{v}\|^2_{L^2(0,T;H^2)} + \|\nabla\bar{\sigma}\|^2_{L^2(0,T;H^2)}],
$$

<u>for every positive</u> ε <u>verifying</u>

$$
(2.13) \qquad \varepsilon < \frac{\lambda}{2c_1},
$$

<u>where</u> c_1 <u>is the constant in</u> (2.16). <u>Here</u> c_1, c_2, c_3 <u>are positive constants depending only on</u> Ω.

<u>Proof</u>. The existence of a solution σ in the required space follows with standard techniques from the a priori bound (2.12), or from [9], Volume II, Chapter 4, Theorem 5.2, with $H = H^1$. Let us prove (2.12):

By application of the operator Λ to both sides of $(2.11)_3$, by multiplication by $\Delta\sigma$, and by integration over Ω one gets $\frac{1}{2} D_t \|\Delta\sigma\|^2 + (\nabla(\lambda\Delta\sigma - \bar{v}\cdot\nabla\bar{\sigma}), \nabla\Delta\sigma) = 0$. Hence

$$(2.14)\qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\Delta\sigma\|^2 \leq$$
$$\leq (\|D\bar{v}D\bar{\sigma}\| + \|\bar{v}D^2\bar{\sigma}\|)\,\|\nabla\Delta\sigma\| .$$

By using Sobolev's embeding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality it follows that

$$(2.15)\qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\Delta\sigma\|^2 \leq$$
$$\leq c(\|D\bar{v}\|^{1/2}\|D\bar{v}\|_1^{1/2}\|\nabla\bar{\sigma}\|_1 +$$
$$+ \|\bar{v}\|_1\|\nabla\bar{\sigma}\|_1^{1/2}\|\nabla\Delta\bar{\sigma}\|^{1/2})\quad \|\nabla\Delta\sigma\| .$$

An utilization of $abc \leq (8\varepsilon^{-3})a^4 + (\varepsilon/2)\,b^2 + (\varepsilon/2)\,c^4$, $\varepsilon > 0$, leads to

$$(2.16)\qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\Delta\sigma\|^2 \leq c\,\varepsilon\|D\bar{v}\|_1^2 +$$
$$+ c_1\,\varepsilon\|\nabla\Delta\sigma\|^2 + \frac{c}{\varepsilon^3}\|\bar{v}\|_1^4\,\|\nabla\bar{\sigma}\|_1^2 +$$
$$+ c\,\varepsilon\,\|\nabla\Delta\bar{\sigma}\|^2 + \frac{c}{\varepsilon^3}\|\bar{v}\|_1^2\,\|\nabla\bar{\sigma}\|_1^4 .$$

Hence for ε verifying (2.13) one has

$$(2.17)\qquad \frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\,\|\nabla\Delta\sigma\|^2 \leq \frac{c}{\varepsilon^3}(\|\bar{v}\|_1^2\,\|\nabla\bar{\sigma}\|_1^4 +$$
$$+ \|\bar{v}\|_1^4\,\|\nabla\bar{\sigma}\|_1^2) + c_9\varepsilon\,(\|D\bar{v}\|_1^2 + \|\nabla\Delta\bar{\sigma}\|^2),$$

where the constants c depend only on Ω. This proves inequality (2.12). Recall that $\|\sigma\|_2 \leq c\|\Delta\sigma\|$, $\|\sigma\|_3 \leq c\|\nabla\Delta\sigma\|$.

<u>Remark</u>. One could also consider the linearized equation $D_t\,\sigma + \bar{v}\cdot\nabla\sigma - \lambda\Delta\sigma = 0$ instead of $(2.11)_3$; then estimate (2.17) holds with $\bar{\sigma}$ replaced by σ and without the term $c_9\varepsilon\,\|\nabla\Delta\bar{\sigma}\|^2$. In this case the solution σ of the linearized problem verifies the maximum principle (which doesn't hold for the solution of $(2.11)_3$). However, the linearization $(2.11)_3$ seems to be more in keeping with the linearization $(2.11)_1$. Besides, the maximum principle will be recovered for the solution of the full nonlinear problem $(2.10)_3$.

3. <u>The nonlinear problem. Local existence</u>. We will not take care of the explicit dependence on μ,λ,m,M; some of the constants c, c_k, depend on these fixed quantities. In order to simplify the equations, we denote by K_o, K_1, K_2,..., constants depending on the norms of the initial data $\|v_o\|_V$ and $\|\sigma_o\|_2$.

In this section we solve (2.10) by proving the existence of a fixed point $(\rho,v) = (\bar\rho,\bar v)$ for system (2.11). Define

$$K_1 \equiv \{\bar v : \bar v_{|t=0} = v_o(x),\ \|\bar v\|^2_{L^2(0,T;H^2)} + \|\bar v\|^2_{C(0,T;V)} + \|\bar v'\|^2_{L^2(0,T;H)} < 2\, C_4 \|v_o\|^2_V \}$$

$$K_2 \equiv \{\bar\sigma : \bar\sigma_{|t=0} = \sigma_o(x),\ \|\bar\sigma\|^2_{L^2(0,T;H^3_N)} + \|\bar\sigma\|^2_{C(0,T;H^2_N)} < 2\, c_2 \|\sigma_o\|^2_2,\ \|\sigma'\|_{L^2(0,T;H^1)} < K_o,$$

$$\|\bar\sigma - \sigma_o\|_{C(\bar Q_T)} < \frac{m}{2} \} ,$$

where $C_4 \equiv \mu[\min\{\mu, M, (m\mu^2)/(4M^2)\}]^{-1}$ and $K_o \equiv \lambda\sqrt{2c_2}\,\|\sigma_o\|_2 + \bar c\,\sqrt{4c_2c_4}\,\|v_o\|_V \|\sigma_o\|_2$.
Here $\bar c = \bar c(\Omega)$ is a positive constant such that

(3.1) $$\|\bar v\cdot\bar w\|_1 < \bar c\, \|\bar v\|_V \|\bar w\|_2 ,\ \forall\, \bar v \in V,\ \bar w \in H^2 .$$

Note that for every $\bar\sigma \in K_2$ one has in Q_T

(3.2) $$\frac{m}{2} < \bar\rho(t,x) < M + \frac{m}{2} .$$

We now evaluate the L^2 norm of $\bar F \equiv F(\bar\rho,\bar v)$. By using Sobolev's embeding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality one easily gets

(3.3) $$\|F(\bar\rho,\bar v)\|^2 < C\, \|\bar v\|_1^3\ \|D\bar v\|_1 + C\, \|\bar v\|_1^2\, \|D^2\bar\sigma\| \cdot \|D^2\bar\sigma\|_1 + C\, \|\bar v\|_1\, \|D\bar v\|_1\, \|D\bar\sigma\|_1^2 + C\, \|D\bar\sigma\|_1^3\ \|D^2\bar\sigma\|_1 + C\, \|D\sigma\|_1^6 + C\, \|f\|^2 .$$

Consequently,

(3.4) $$\|F(\bar\rho,\bar v)\|^2_{L^2(0,T;H)} < C\, (\|v_o\|_V + \|\sigma_o\|_2)^{7/2}\ T^{1/2} + C\, \|\sigma_o\|_2^6\ T + C\, \|f\|^2_{L^2(0,T;H)} .$$

Hence, by using (2.3), it follows that the solution v of $(2.11)_1$, $(2.11)_2$ verifies

(3.5) $\|v\|^2_{C(0,T;V)} + \|v\|^2_{L^2(0,T;H^2)} + \|v'\|^2_{L^2(0,T;L^2)} <$

$$< C_4 \|v_o\|^2_V + K_1(\sqrt{T} + T) + C \|f\|^2_{L^2(0,T;H)} .$$

On the other hand, from (2.12),

(3.6) $$\|\sigma\|^2_{C(0,T;H^2_N)} + \|\sigma\|^2_{L^2(0,T;H^3_N)} <$$

$$< \{ C_2 \|\sigma_o\|^2_2 + K_3 \varepsilon^{-3} T + K_4 \varepsilon \} .$$

Now we fix $\varepsilon > 0$ such that $K_4 \varepsilon < 2^{-1} C_2 \|\sigma_o\|^2_2$. Finally, by fixing a sufficiently small $T > 0$, it follows that $v \in K_1$, $\sigma \in K_2$. The estimate for $D_t \sigma$ follows by using equation $(2.11)_3$ and (3.1). The estimate for the sup norm of $\sigma - \sigma_o$ in $\bar{Q}_T$ is proved as follows:

Clearly, $\|\sigma(t) - \sigma_o\|_1 < \int_o^t \|\sigma'(s)\|_1 \, ds < K_o T^{1/2}$. On the other hand $\|\sigma(t) - \sigma_o\|_{C(\bar{\Omega})} < C_5 \|\sigma(t)-\sigma_o\|_1^{1/3} \|\sigma(t)-\sigma_o\|_2^{2/3}$, where C_5 depends only on Ω; recall that $H^{5/3}(\Omega) \hookrightarrow C(\bar{\Omega})$. Consequently

$$\|\sigma - \sigma_o\|_{C(\bar{Q}_T)} < C_5 K_o^{1/3} T^{1/6} (C_6 \sqrt{2C_2} \|\sigma_o\|_2 + \|\sigma_o\|_2)^{2/3} ,$$

where $C_6 = C_6(\Omega)$ is a positive constant, such that $\|\sigma\|_{C(\bar{\Omega})} < C_6 \|\sigma\|_2$, $\forall \sigma \in H^2_N(\Omega)$. Hence, by choosing (if necessary) a smaller value for T, one gets $\|\sigma - \sigma_o\|_{C(\bar{Q}_T)} < m/2$.

Now we utilize Shauder's fixed point theorem. Clearly $K \equiv K_1 + K_2$ is a convex, compact set in $L^2(0,T;L^2) \times L^2(0,T;L^2)$. Let us denote by Φ the map $\Phi(\bar{\rho},\bar{v}) = (\rho,v)$, defined by (2.11). Since $\Phi(K) \subset K$, it is sufficient to prove that $\Phi : K \to K$ is continuous in the L^2 topology. If $\bar{v}_n \to \bar{v}$ in $L^2(Q_T)$, $\bar{\rho}_n \to \bar{\rho}$ in $L^2(Q_T)$, is follows by compactness arguments that $\bar{v}_n \rightharpoonup \bar{v}$ weakly in $L^2(0,T;H^2)$ and in $H^1(0,T;L^2)$, and that $\nabla\bar{\rho}_n \to \nabla\bar{\rho}$ weakly in $L^2(0,T;H^2)$ and in $H^1(0,T;L^2)$. In particular $\bar{\rho}_n$ is a bounded sequence in $H^{1/2+\varepsilon_1}(0,T;H^{2-\varepsilon_2}) \hookrightarrow C^{o,\alpha}(\bar{Q}_T)$ (3), for suitable positive ε_1, ε_2, α.

(3) α-Hölder continuous functions in $\bar{Q}_T$.

Hence $\bar{\rho}_\nu \to \bar{\rho}$ uniformly in $\bar{Q}_T$. Moreover, $\bar{v}_n$ and $\nabla\bar{\rho}_n$ are bounded in $H^{1/2}(0,T;H^1)$ and in $H^{1/2}(0,T;H^1)$ respectively. Hence $\bar{v}_n \to \bar{v}$ and $\nabla\bar{\rho}_n \to \nabla\bar{\rho}$ strongly in the L^4 topology. It follows from (2.8) at $F(\bar{\rho}_n, \bar{v}_n) \to F(\bar{\rho},\bar{v})$ as a distribution in Q_T; consequently $F(\bar{\rho}_n, \bar{v}_n) \to F(\bar{\rho},\bar{v})$ weakly in $L^2(Q_T)$, because $F(\bar{\rho}_n,\bar{v}_n)$ is a bounded sequence in this space. Analogously, $\bar{v}_n \cdot \nabla\bar{\rho}_n \to \bar{v} \cdot \nabla\bar{\rho}$ strongly in $L^2(Q_T)$. It follows from the linear equations (2.11) that $v_n \to v$ and $\rho_n \to \rho$ in $L^2(Q_T)$ and $L^2(Q_T)$, respectively. Hence Φ is continuous. This finishes the proof of the existence of a local solution. Uniqueness will be proved in Section 5.

4. Global solutions. Asymptotic behavior.

In this section the constants C_K depend at most on Ω and on the quantities μ, λ and $\hat{\rho}$, i.e. on the total amount of mass $|\Omega|\,\hat{\rho}$. We assume that

$$\|\sigma_o\|_2 < (2\bar{C})^{-1}\,\hat{\rho}, \tag{4.1}$$

hence $\hat{\rho}/2 < m < M < 3\,\hat{\rho}/2$. Let (ρ,v) be a solution of (1.1). From (2.6) for $\varepsilon_o = (4M^2)^{-1} m\,\mu$, and from (3.3) one gets

$$\frac{m}{2}\|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \frac{m\mu^2}{8\,M^2}\|Av\|^2 < \tag{4.2}$$

$$< C\,(\|v\|_1^2 + \|\sigma\|_2^3)\,(\|v\|_2 + \|\sigma\|_3) + C\,\|\sigma\|_2^6 + C\,\|f\|^2\,,$$

where C depends only on Ω, μ, $\hat{\rho}$. On the other hand, from (2.17) for $\varepsilon = (2C_9)^{-1}\lambda$, one gets

$$\frac{d}{dt}\|\Delta\sigma\|^2 + \frac{\lambda}{2}\|\nabla\Delta\sigma\|^2 < C\,(\|v\|_1^6 + \|\sigma\|_2^6)\,. \tag{4.3}$$

From (4.2) and (4.3) it easily follows that

$$\frac{d}{dt}\Big(\frac{\mu}{2}\|v\|_V^2 + \|\Delta\sigma\|^2\Big) + \frac{m}{2}\|D_t v\|^2 +$$

$$+ \frac{m\mu^2}{16M^2}\|Av\|^2 + \frac{\lambda}{4}\|\nabla\Delta\sigma\|^2 < C(\|v\|_V^6 + \|\Delta\sigma\|^6 + \|f\|^2)\,.$$

In particular, since $\|Av\| \geq C\|v\|_V$ and $\|\nabla\Delta\sigma\| \geq C\|\Delta\sigma\|$, one has

$$\frac{d}{dt}(\|v\|_V^2 + \|\Delta\sigma\|^2) \leq -[C_{10} - C_{11}(\|v\|_V^2 + \|\Delta\sigma\|^2)^2] \cdot \cdot (\|v\|_V^2 + \|\Delta\sigma\|^2) + C_{12}\|f\|^2. \tag{4.4}$$

Hence $(4.5)_1$ below holds for every $t \in [0,+\infty[$ if

$$\begin{cases} C_{11}(\|v_o\|^2 + \|\Delta\sigma_o\|^2)^2 < \dfrac{C_{10}}{2}\, m \\[2ex] C_{12}\,\|f\|^2_{L^\infty(0,+\infty;H)} < \dfrac{C_{10}}{2}\sqrt{\dfrac{C_{10}}{2C_{11}}}\,. \end{cases} \tag{4.5}$$

In fact, if $C_{11}(\|v(t)\|_V^2 + \|\Delta\sigma(t)^2\|)^2 = C_{10/2}$ it must be, from (4.4), that $\frac{d}{dt}(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2) < 0$.

Let us now prove the last assertion in theorem A. Under the hypothesis $(4.5)_1$, it follows from (4.4) that

$$\frac{d}{dt}(\|v\|_V^2 + \|\Delta\sigma\|^2) \leq -\frac{C_{10}}{2}(\|v\|_V^2 + \|\Delta\sigma\|^2)\,.$$

this proves (1.6) . □

5. Uniqueness. We prove that the solution (ρ,v) of problem (1.1) is unique in the class in which existence was proved; see Theorem A. We remark that more careful calculations lead to uniqueness in a larger class.

Let (ρ,v), $(\bar\rho,\bar v)$ be two solutions of problem (1.1), (1.2) and put $u = v-\bar v$, $\eta = \rho-\bar\rho$. By subtracting the equations $(2.10)_1$ written for (ρ,v) and $(\bar\rho,\bar v)$ respectively, and by taking the inner product with u in H one gets

$$\frac{1}{2}\frac{d}{dt}(\rho u,u) + \mu\,\|u\|_V^2 = -\frac{1}{2}(v\cdot\nabla\rho,u^2) + + \frac{\lambda}{2}(\Delta\rho,u^2) - (\mu, D_t\bar v\cdot u) + (F-\bar F,u)\,.$$

By using $(\Delta\rho,u^2) = -(\nabla\rho,Du^2)$, we show that

$$\frac{1}{2}\frac{d}{dt}(\rho u,u) + \frac{\mu}{2}\|u\|_V^2 \leq \frac{1}{2}\|v\|_\infty\|\nabla\rho\|_\infty\|u\|^2 + + \frac{\lambda^2}{\mu}\|\nabla\rho\|_\infty^2\,\|u\|^2 + C\|D_t\bar v\|^2\,\|u\|^2 + + \frac{\lambda}{4}\|\Delta\eta\|^2 + (F-\bar F,u)\,. \tag{5.1}$$

On the other hand, by subtracting equations $(2.10)_3$ written for (ρ,v) and for $(\bar{\rho},\bar{v})$ respectively, and by taking the inner product with $\Delta\eta$ in $L^2(\Omega)$ one gets

$$\frac{1}{2}\frac{d}{dt}\|\nabla\eta\|^2 + \frac{\lambda}{2}\|\Delta\eta\|^2 \le c\,\|\nabla\bar{\rho}\|_\infty^2\,\|u\|^2 + c\,\|v\|_\infty^2\,\|\nabla\eta\|^2 . \tag{5.2}$$

By adding (5.1) and (5.2) it follows that

$$\frac{d}{dt}[(\rho u,u) + \|\nabla\eta\|^2] + \mu\,\|u\|_V^2 + \frac{\lambda}{2}\|\Delta\eta\|^2 \le \theta_1(t)\left(\|u\|^2 + \|\nabla\eta\|^2\right) + (F-\bar{F},u), \tag{5.3}$$

where $\theta_1(t)$ is a real integrable function on $[0,T]$.

On the other hand, by using Sobolev's embeeding theorems and Hölder's inequality (and also $ab \le \varepsilon a^2+\varepsilon^{-1}b^2$) the reader easily verifies that given $\varepsilon > 0$ there exists an integrable real function $\theta_2(t)$ (dependent on ρ, $\bar{\rho}$, v, $\bar{v}$ and on ε) such that

$$|(F-\bar{F},u)| \le \theta_2(t)\,\|u\|^2 + \varepsilon\,(\|\eta\|_2^2+\|u\|_1^2) . \tag{5.4}$$

By using $\|u\|^2 \le m^{-1}(\rho u,u)$, (5.3) and (5.4) it follows that

$$\frac{d}{dt}[(\rho u,u) + \|\nabla\eta\|^2] \le (\theta_1(t) + \theta_2(t))\,[(\rho u,u) + \|\nabla\eta\|^2)].$$

Uniqueness follows now from Gronwall's lemma and from $u_{|t=0} = 0$, $\eta_{|t=0} = 0$. $\square$

REFERENCES

1. H. Beirão-da-Veiga, R. Serapioni, A. Valli "On the Motion of Non-homogeneous fluids in the Presence of Diffusion", J. Math. Anal. Appl., 85 (1982), 179-191.

2. D.A. Frank-Kamenetskii "Diffusion and Heat Transfer in chemical Kinetics", Plenum ed., New York, London (1969).

3. E. Gagliardo "Caretterizzazioni delle tracce sulla Frontiera Relative ad alcune classi di Functioni in M Variabili", Rend. Sem. Mat. Univ. Padova, 27 (1957), 284-305.

4. V.N. Ignat'ev, B.G. Kuznetsov "A Model for the Diffusion of a Turbulent Boundary Layer in a Polymer", Čisl. Metody Meh. Splošn. Sredy, 4 (1973), 78-87 (in russian).

5. A.V. Kazhikhov, Sh. Smagulov, "The Correctness of Boundary-Value Problems in a Diffusion Model of an Inhomogeneous Liquid", Sov. Phys. Dokl. , 22 (1977), 249-250.

6. A.V. Kazhikhov, Sh. Smagulov, "The Correctness of Boundary-Value Problems in a Certain Diffusion Model of an Inhomogeneouse Fluid", Čisl. Metody Meh. Splošn. Sredy, 7 (1976), 75-92.

7. O.A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow", Gordon and Breach, New York (1969).

8. J.L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires", Dunod, Gauthier-Villars, Paris (1969).

9. J.L. Lions, E. Magenes "Problèmes aux limites nonhomogènes et applications", Vol. I,II, Dunod, Paris (1968).

10. G. Prouse, A. Zaretti "On the Inequalities Associated to a Model of Graffi for the Motion of a Mixture of two viscous Incompressible Fluids", preprint Departimento di Matematica - Politecuico di Milano (1982).

11. P. Secchi "On the Initial Value Problem for the Equations of Motion of Viscous Incompressible Fluids in Presence of Diffusion", Boll. U.M.I., 1-B(1982) 1117-1130.

12. R. Temam "Navier-Stokes Equations", North Holland Amsterdam (1977).

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We consider the motion of a mixture of two fluids, with a diffusion effect obeying Fick's law; for the derivation of the model (1.1) see Section 1 and references [2], [4], [5] and [6]. We consider the full non-linear problem (i.e., we don't omit the term λ^2 term in equation (1.1_1). Moreover, we don't assume that λ/μ is small. We prove the existence of a (unique) local

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ABSTRACT (Continued)

solution, the existence of a global solution for small data, and the exponential decay to the equilibrium solution; see Theorem A, Section 1.

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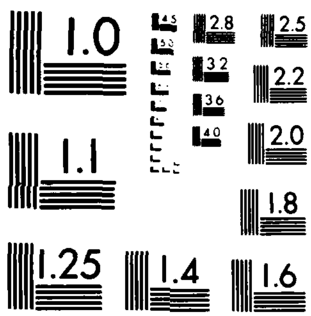

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DIFFUSION ON VISCOUS FLUIDS, EXISTENCE AND ASYMPTOTIC PROPERTIES OF SOLUTIONS

H. Beirão-da-Veiga

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

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ABSTRACT

~~We~~ The document considers the motion of a mixture of two fluids, with a diffusion effect obeying Fick's law; for the derivation of the model (1.1) see Section 1 and references [2], [4],[5] and [6]. ~~We~~ The author considers the full non-linear problem (i.e., we don't omit the term λ^2 term in equation (1.1_1). Moreover, we ~~don't~~ and does not assume that ~~λ/μ~~ lambda/micron is small. ~~We~~ He proves the existence of a (unique) local solution, the existence of a global solution for small data, and the exponential decay to the equilibrium solution; see Theorem A, Section 1.

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*Department of Mathematics, University of Trento, 38050 Povo (Trento) Italy.

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SIGNIFICANCE AND EXPLANATION

In this paper we consider the motion of a continuous medium consisting of two components, say water and a dissolved salt, with a diffusion effect obeying Fick's law. We denote by $v, w, \rho, \pi, \mu, \lambda$ the mean-volume velocity, mean-mass velocity, density, pressure, viscosity and diffusion constant, respectively. By using Fick's law we eliminate w from the equations and we obtain (1.1), where p is the modified pressure; see Section 1 and references [2], [4], [5] and [6]. The initial boundary conditions are given by equation (1.2).

Kazhikhov and Smagulov [5],[6] consider equation (1.1) for a small diffusion coefficient λ. More precisely, they assume that condition (1.3) holds; moreover, they omit the λ^2 term in equation $(1.1)_1$. Under these conditions they prove the existence of a unique local solution for the 3-dimensional motion (in the bi-dimensional case, solutions are global).

In our paper we consider the full equation (1.1), without assumption (1.3), and we prove: (i) the existence of a (unique) local solution; (ii) the existence of a global solution in time for small initial velocities and external forces, and for initial densities near-constant; (iii) the exponential decay (when $t \to +\infty$) of the solution (ρ, v) to the equilibrium solution $(\hat{\rho}, 0)$, if $f \equiv 0$. See Theorem A, Section 1.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DIFFUSION ON VISCOUS FLUIDS, EXISTENCE
AND ASYMPTOTIC PROPERTIES OF SOLUTIONS

H. Beirão-da-Veiga*

Main Notation

Ω : an open bounded set in R^3, locally situated on one side of its boundary Γ, a regular (say C^4) manifold.

$n = n(x)$: unit outward normal to Γ.

D_i, D_{ij}, D_t : $\partial/\partial x_i$, $\partial^2/\partial x_i \partial x_j$, $\partial/\partial t$.

$\| \ \|$, $(\ , \ )$: norm and scalar product in $L^2(\Omega)$.

H^k : Sobolev space $H^{k,2}(\Omega)$ with norm

$$\|\sigma\|_k^2 \equiv \sum_{l=0}^{k} \|D^l\sigma\|^2,$$

where

$$\|D^l\sigma\|^2 \equiv \sum_{|\alpha|=l} \|D^\alpha\sigma\|^2,$$

Further,

$$\|D^l\sigma\|_m^2 \equiv \sum_{|\alpha|=l} \|D^\alpha\sigma\|_m^2 .$$

H_0^1 : Closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

$\| \ \|_\infty$: norm in $L^\infty(\Omega)$.

L^2, H^k, H_0^1 : Hilbert spaces of vectors $v = (v_1, v_2, v_3)$ such that $v_i \in L^2, v_i \in H^k, v_i \in H_0^1$ $(i=1,2,3)$, respectively. Corresponding notation is used for other spaces of vector fields. Norms are defined in the natural way, and denoted by the symbols used for the scalar fields.

Let us introduce the following functional spaces (see, for instance, [7], [8] and [12] for their properties) -

*Department of Mathematics, University of Trento, 38050 Povo (Trento) Italy.

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$$H_N^k \equiv \{\sigma \in H^k : \frac{\partial\sigma}{\partial n} = 0 \text{ on } \Gamma \text{ and } \int_\Omega \sigma(x)dx = 0\},\ k > 2\ .$$

$$\nu \equiv \{ v \in C_o^\infty(\Omega) : \text{div } v = o \text{ in } \Omega\},$$

$$H = \{ v \in L^2 : \text{div } v = 0 \text{ in } \Omega,\ v\cdot n = 0 \text{ on } \Gamma\},$$

$$V = \{ v \in H_0^1 : \text{div } v = 0 \text{ on } \Omega\}.$$

H and V are the closure of ν in $L^2(\Omega)$ and H_o^1, respectively. Moreover $L^2 = H + G$, where $G \equiv \{ \nabla p : p \in H^1(\Omega)\}$. Denoting by P the orthogonal projection of L^2 onto H, we define the operator $A \equiv -P\Delta$ on $D(A) \equiv H^2 \cap V$. One has $(Au,v) = ((u,v)) \equiv \sum_{i,j} (D_i U_j, D_i v_j)$, $\forall\, u \in D(A)$, $v \in V$.

The norms $\|\sigma\|_2$, $\|\Delta\sigma\|$ are equivalent in H_N^2, $\|\sigma\|_3$, $\|\nabla\Delta\sigma\|$ are equivalent in H_N^3 and $\|v\|_2$, $\|Av\|$ are equivalent in $D(A)$. We define $\|v\|_V^2 \equiv ((v,v))$; the norms $\|v\|_V$, $\|v\|_1$ are equivalent in V.

$L^2(0,T;X)$: Banach space of strongly measurable functions defined in $]0,T[$ with values in (a Banach space) X, for which

$$\|z\|^2_{L^2(0,T;X)} \equiv \int_0^T \|z(t)\|_X^2\, dt < +\infty.$$

$C(0,T;X)$: Banach space of X-vector valued continuous functions on $[0,T]$ endowed with the usual norm $\|z\|_{C(0,T;X)}$.

μ : viscosity (a positive constant).

λ : diffusion coefficient (a positive constant).

$v(t,x)$, $v_o(x)$: mean-volume velocity. Initial mean-volume velocity.

$\rho(t,x)$, $\rho_o(x)$: density of the mixture. Initial density.

Further,

$$m \equiv \inf_{x\in\Omega} \rho_o(x)\ ,\quad M \equiv \sup_{x\in\Omega} \rho_o(x),$$

$$\hat{\rho} \equiv \frac{1}{|\Omega|} \int_\Omega \rho_o(x)dx.$$

We assume that $m > 0$.

$\pi(t,x)$, $p(t,x)$: pressure. Modified pressure

$$p = \pi + \lambda\, v \cdot\nabla\rho - \lambda^2\Delta\rho + \lambda(2\mu+\mu')\Delta \log \rho.$$

$f(t,x)$: external mass-force.

We denote by c, $\bar{c}$, c_o, c_1, c_2, ... positive constant, depending at most on Ω and on the parameters μ, λ, m, M and $\hat{\rho}$. It is easy to derive at any stage of the proofs, the explicit dependence of the constants on the parameters.

For convenience we sometimes denote different constants by the same symbol c. Otherwise, we utilize the symbols $\bar{c}$, c_k, $k \in \mathbb{N}$.

1. Main results. In this paper we consider the motion of a viscous fluid consisting of two components, say, saturated salt water and water. The equations of the model are obtained, for example, in [2], [4], [5], and [6]. Let us give a brief sketch. Let ρ_1, ρ_2 be the characteristic densities (constants) of the two components, $v^{(1)}(t,x)$ and $v^{(2)}(t,x)$ their velocities, and $e(t,x)$, $d(t,x)$ the mass and volume concentration of the first fluid. We define the density $\rho(t,x) \equiv d\rho_1 + (1-d)\,\rho_2$, and the mean-volume and mean-mass velocities $v \equiv d\,v^{(1)} + (1-d)\,v^{(2)}$, $w \equiv e\,v^{(1)} + (1-e)v^{(2)}$. Then the equations of motion are

$$\begin{cases} \rho[D_t w + (w\cdot\nabla)w - f] - \mu\Delta w - (\mu+\mu')\nabla\,\mathrm{div}\, w = -\nabla\pi, \\ \mathrm{div}\, v = 0\,, \\ D_t\rho + \mathrm{div}\,(\rho w) = 0\,. \end{cases}$$

On the other hand, Fick's diffusion law (see [2]) gives $w = v - \lambda\rho^{-1}\nabla\rho$. By elimination of w in the preceeding equations one gets, after some calculations,

$$(1.1)\qquad \begin{cases} \rho(D_t v + (v\cdot\nabla)v) - \mu\Delta v - \lambda\,[(v\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)v] + \\ \quad + \dfrac{\lambda^2}{\rho}\,[(\nabla\rho\cdot\nabla)\nabla\rho - \dfrac{1}{\rho}\,(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho] = -\,\nabla p + \rho f, \\ D_t\rho + v\nabla\rho - \lambda\Delta\rho = 0\,, \\ \mathrm{div}\, v = 0\,. \end{cases}$$

We want to solve system (1.1) in $Q_T \equiv]0,T[\times \Omega$. Here p is the modified pressure. We add to system (1.1) the following initial boundary-value conditions.

$$(1.2)\quad \begin{cases} v = 0 & \text{on }]0,T[\times \Gamma, \\ \dfrac{\partial\rho}{\partial n} = 0 & \text{on }]0,T[\times \Gamma, \\ v_{|t=0} = v_o(x) & \text{in } \Omega, \\ \rho_{|t=0} = \rho_o(x) & \text{in } \Omega . \end{cases}$$

The first two conditions mean that there is no flux through the boundary.

In [5], [6] Kazhikhov and Smagulov consider the simplified system obtained from (1.1) by omitting the term containing λ^2; moreover, they assume that

$$(1.3)\qquad \lambda < \frac{2\mu}{M-m} .$$

Under these conditions Kazhikhov and Smagulov state the existence of a local solution in time (global in the bidimensional case).

In our paper we take into account the full equation (1.1), and omit the condition (1.3). For this more general case we prove: (i) the existence of a (unique) local solution for arbitrary initial data and external face field; (ii) the existence of a unique global strong solution, for small initial data and external force field. Moreover, if $f \equiv 0$, the solution (ρ,v) decays exponentially to the equilibrium solution $(\hat{\rho},0)$. More precisely we prove the following result:

Theorem A. Let $v_o \in V, \rho_o - \hat{\rho} \in H^2_N$, $f \in L^2(0,T; L^2)$. Then there exists $T_1 \in]0,T]$

such that problem (1.1), (1.2) is uniquely solveable in Q_{T_1}. Moreover $v \in L^2(0,T_1; H^2) \cap C\,(0,T_1; V)$, $D_t v \in L^2\,(0,T_1; H)$, $\rho-\hat{\rho} \in L^2(0,T_1;H^3_N) \cap C\,(0,T_1; H^2_N)$. $D_t\rho \in L^2(0,T_1; H^1)$ and $m \leq \rho(t,x) \leq M$.

Moreover, there exist positive constants k_1, k_2, and k_3 depending at most on Ω, μ, λ and on the mean density $\hat{\rho}$ (1) such that that if

$$(1.4)\qquad \|v_o\|_1 + \|\rho_o-\hat{\rho}\|_2 \leq k_1,$$

(1) Or, equivalently, depending on the total amount of mass $|\Omega|\,\hat{\rho} = \int_\Omega \rho_o(x)dx$.

and

(1.5) $$\|f\|_{L^\infty(0,+\infty;L^2)} \le k_2,$$

<u>then the solution is global in time</u>. <u>If</u> $f \equiv 0$, <u>the solution</u> (ρ,v) <u>decays exponentially to the equilibrium solution</u> $(\hat{\rho},0)$, <u>i.e.</u>

(1.6) $$\|v(t)\|_1 + \|\rho(t)-\hat{\rho}\|_2 \le (\|v_o\|_1+\|\rho_o-\hat{\rho}\|_2)\, e^{-k_3 t},$$

<u>for every</u> $t > 0$.

Theorem A also holds for coefficients μ,λ regularly dependent on ρ, v, provided they are strictly positive and bounded, in a neighborhood of the range of values of the initial data $\rho_o(x)$, $v_o(x)$. This generalization can be done without any difficulty. Moreover, with standard techniques, one can prove that the solutions have more regularity (up to C^∞) if the data are sufficiently regular and the usual compatibility conditions hold.

Local existence in the general case (i.e. with the λ^2 term and without (1.3)) was proved in the <u>inviscid</u> case by Beirão-da-Veiga, Serapioni and Valli in [1]. a similar result, in the viscous case and for $\Omega = \mathbb{R}^3$, was proved by Secchi [11]. For another approach (concerning Greffi's model) see [10].

2. <u>The linearized equations</u>. We start by proving the following theorem: <u>Theorem 2.1</u> <u>Let</u> $\rho(t,x)$ <u>be a measurable function verifying</u>

(2.1) $$0 < m \le \rho(t,x) \le M \quad , \quad \text{a.e., in } Q_T, \text{ let } F \in L^2(0,T;H) \text{ and } v_o \in V.$$

<u>Then there exists a (unique) strong solution</u> v <u>of problem</u>

(2.2) $$\begin{cases} \rho\, D_t v - \mu\Delta v = -\nabla p + F & \text{in } Q_T, \\ \operatorname{div} v = 0 & \text{in } Q_T, \\ v = 0 & \text{on }]0,T[\times \Gamma, \\ v_{|t=0} = v_o(x) & \text{in } \Omega. \end{cases}$$

<u>Moreover</u> $v \in L^2(0,T;D(A)) \cap C(0,T;V)$, $D_t v \in L^2(0,T;H)$ <u>and</u>

(2.3) $$\mu\|v\|^2_{C(0,T;V)} + m\,\|v'\|^2_{L^2(0,T;H)} + \frac{m\mu^2}{4M^2}\,\|Av\|^2_{L^2(0,T;H)} \le \mu\|v_o\|^2_V + \left(\frac{2}{m} + \frac{m}{2M^2}\right)\|F\|^2_{L^2(0,T;H)}$$

Proof. Let us write equation (2.2) in the equivalent form(2.4)

$P(\rho D_t v) + \mu A v = F$, $v_{|t=0} = v_o(X)$.

For brevity let us put

$$X = \{v\colon v \in L^2(0,T;D(A)),\ v' \in L^2(0,T;H)\} .$$

From well known results (see [9], Vol I, Chapter I : Theorem 3.1 with y = H, X = D(A), j = 0; and (2.42) Proposition 2.1) it follows that $X \hookrightarrow C(0,T;V)$.

We start by proving the a priori bound (2.3); an essential device is to introduce a parameter ε_o in order to conveniently balance the estimates. In H take the inner product of (2.4) with $D_t v + \varepsilon_o A v$, $\varepsilon_o > 0$. Since $(v', Av) = 2^{-1} D_t\|v\|_V^2$ one gets

$$\text{(2.5)} \qquad m \|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \varepsilon_o \mu \|Av\|^2 <$$
$$< \|F\|\,\|D_t v\| + \varepsilon_o\|F\|\,\|Av\| + \varepsilon_o M\,\|D_t v\|\,\|Av\| .$$

By using the inequalities $\|F\|\,\|D_t v\| < 4^{-1}m\,\|D_t v\|^2 + m^{-1}\,\|F\|^2$, $\|F\|\,\|Av\| < 4^{-1}\mu\,\|Av\|^2 + \mu^{-1}\,\|F\|^2$ and $\|D_t v\|\|Av\| < (4M)^{-1}\mu\,\|Av\|^2 + \mu^{-1}M\,\|D_t v\|^2$ one gets

$$\text{(2.6)} \qquad \frac{3}{4}\, m\,\|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \varepsilon_o\frac{\mu}{2}\|Av\|^2 <$$
$$< \left(\frac{1}{m} + \frac{\varepsilon_o}{\mu}\right)\|F\|^2 + \frac{\varepsilon_o M^2}{\mu}\|D_t v\|^2 .$$

Now fix $\varepsilon_o = (4M^2)^{-1}m\mu$ and integrate equation (2.6) on (0,T). This gives the a priori bound (2.3).

Define $\|v\|_X^2 \equiv$ "left hand side of equation (2.3)", $Y \equiv L^2(0,T;H) \times V$, and $\|(F,v_o)\|_Y^2 \equiv$ "right hand side of (2.3)".

We solve (2.4) by the continuity method. Define $\rho\alpha \equiv (1-\alpha)\,\hat{\rho} + \alpha\rho$, $\alpha \in [0,1]$. Clearly ρ_α verifies condition (2.1), for any α. Define $T_\alpha \equiv (1-\alpha)\bar{T} + \alpha T$, where

$$\underline{T}v \equiv (P(\rho D_t v) - \mu A v,\ v_{|t=0}) \in Y,$$
$$\bar{T}v \equiv (P(\bar{\rho} D_t v) - \mu A v,\ v_{|t=0}) \in Y.$$

Finally denote by γ the set of values $\alpha \in [0,1]$ for which problem (2.2) is solvable in X for every pair $(F,v_o) \in Y$. Clearly $o \in \gamma$, because for this value of the parameter equation (2.6) because the linearized Navier-Stokes equation. Let us verify that γ *is open and closed.*

$\underline{\gamma \text{ is open}}$. Let $\alpha_o \in \gamma$ and denote by $G(F,v_o) \equiv v$ the solution v of problem $T_{\alpha_o} v = (F,v_o)$. From (2.3) one gets $G \in L(Y;X)$ ([2]) with $\|G\|_{Y,X} \le 1$. Equation $T_{\alpha_o+\varepsilon} v = (F,v_o)$ can be written in the form

$$[1-\varepsilon\, G(\bar{T}-T)]v = G(F,v_o). \tag{2.7}$$

Since $\|G(\bar{T}-T)\|_{X,Y} \le \|\bar{T}-T\|_{X,Y}$, equation (2.7) is solvable for $|\varepsilon| < \|\bar{T}-T\|^{-1}_{X,Y}$ (by a Neuman expansion).

$\underline{\gamma \text{ is closed}}$. Let $\alpha_m \in \gamma$, $\alpha_m \to \alpha_o$, and let v_n be the solution of $T_{\alpha_n} v_n = (F,v_o)$. From (2.3) one has $\|v_n\|_X \le \|(F,v_o)\|_Y$. Since X is an Hilbert space there exists a subsequence $v_\nu \to v \in X$, weakly in X. From T, $\bar{T} \in L(X;Y)$ one has $\bar{T}v_\nu \to \bar{T}v$, $Tv_\nu \to Tv$ weakly in Y. Hence $T_{\alpha_\nu} v_\nu \to T_{\alpha_o} v$, i.e. $T_{\alpha_o} v = (F,v_o)$.

Let us now return to problem (1.1). Define

$$F(\rho,v) \equiv P\,\{-\rho(v\cdot\nabla)v + \lambda\,[(v\cdot\nabla)\nabla\rho + (\nabla\rho\cdot\nabla)v] + \frac{\lambda^2}{\rho}\,[(\nabla\rho\cdot\nabla)\nabla\rho - \frac{1}{\rho}(\nabla\rho\cdot\nabla\rho)\nabla\rho + \Delta\rho\nabla\rho] + \rho f\}\,. \tag{2.8}$$

For convenience we will use in the sequel the translation

$$\rho = \hat{\rho} + \sigma. \tag{2.9}$$

Recall that $\hat{\rho}$ is a given constant. To solve problem (1.1) and (1.2) in our functional framework is equivalent to finding $v \in L^2(0,T;D(A))$, $v_t \in L^2(0,T;H)$ and $\sigma \in L^2(0,T;H^3_N)$, $\sigma' \in L^2(0,T;H^1)$ such that

$$\begin{cases} P(\rho D_t v) - \mu A v = F(\rho,v), \\ v_{|t=0} = v_o(x), \\ D_t\sigma - \lambda\nabla\sigma = -v\cdot\nabla\sigma, \\ \sigma_{|t=0} = \sigma_o(x), \end{cases} \tag{2.10}$$

([2]) Banach space of linear continuous operator from Y into X, with norm $\|\ \|_{Y,X}$.

where $v_o \in V$ and $\sigma_o(x) - \hat{\rho} \in H^2_N(\Omega)$, are given. Note that from the above conditions on σ it follows that $\sigma \in C(0,T; H^2_N)$.

We solve (2.10) by considering the linearized problem

$$
(2.11) \qquad \begin{cases} P(\rho D_t v) - \mu\Delta v = F(\bar{\rho},\bar{v}) \equiv \bar{F}, \\ v_{|t=0} = v_o(x), \\ D_t\sigma - \lambda\Delta\sigma = -\bar{v}\cdot\nabla\bar{\sigma}, \\ \sigma_{|t=0} = \sigma_o(x), \end{cases}
$$

and by proving the existence of a fixed point $(\bar{\rho},\bar{v}) = (\rho,v)$ for the map $(\bar{\rho},\bar{v}) \to (\rho,v)$ defined by (2.11).

In order to get a sufficiently strong estimate for the linearized equation (2.11) we take in account the particular form of the date $\bar{v}\cdot\nabla\bar{\sigma}$. As for estimate (2.3) we will introduce a balance parameter $\varepsilon > 0$.

<u>Theorem 2.2</u> <u>Assume that</u> $\bar{v} \in L^2(0,T;H^2) \cap C(0,T;H^1_0)$ <u>and that</u> $\bar{\sigma} \in L^2(0,T;H^3_N) \cap C(0,T;H^2_N)$. <u>Then the solution</u> $\sigma \in L^2(0,T;H^3_N)$, $\sigma' \in L^2(0,T;H^1)$ <u>of problem</u> $(2.11)_3$, $(2.11)_4$ <u>verifies the estimate</u>

$$
(2.12) \qquad \|\sigma\|^2_{C(0,T;H^2_N)} + \|\sigma\|^2_{L^2(0,T;H^3_N)} \leq
$$

$$
\leq c_2\|\sigma_o\|^2_2 + c_3\,\varepsilon^{-3}T[\|\bar{v}\|^6_{C(0,t;H^2)} + \|\nabla\bar{\sigma}\|^6_{C(0,T;H^1)}]
$$

$$
+ c_3\,\varepsilon\,[\|\bar{v}\|^2_{L^2(0,T;H^2)} + \|\nabla\bar{\sigma}\|^2_{L^2(0,T;H^2)}],
$$

<u>for every positive</u> ε <u>verifying</u>

$$
(2.13) \qquad \varepsilon < \frac{\lambda}{2c_1},
$$

<u>where</u> c_1 <u>is the constant in</u> (2.16). <u>Here</u> c_1, c_2, c_3 <u>are positive constants depending only on</u> Ω.

<u>Proof</u>. The existence of a solution σ in the required space follows with standard techniques from the a priori bound (2.12), or from [9], Volume II, Chapter 4, Theorem 5.2, with $H = H^1$. Let us prove (2.12):

By application of the operator Δ to both sides of $(2.11)_3$, by multiplication by $\Delta\sigma$, and by integration over Ω one gets $\frac{1}{2} D_t \|\Delta\sigma\|^2 + (\nabla(\lambda\Delta\sigma - \bar{v}\cdot\nabla\bar{\sigma}), \nabla\Delta\sigma) = 0$. Hence

$$(2.14) \qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\Delta\sigma\|^2 \le \le (\|D\bar{v}D\bar{\sigma}\| + \|\bar{v}D^2\bar{\sigma}\|)\,\|\nabla\Delta\sigma\| .$$

By using Sobolev's embeding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality it follows that

$$(2.15) \qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\Delta\sigma\|^2 \le \le c(\|D\bar{v}\|^{1/2}\|D\bar{v}\|_1^{1/2}\|\nabla\bar{\sigma}\|_1 + + \|\bar{v}\|_1\|\nabla\bar{\sigma}\|_1^{1/2}\|\nabla\Delta\bar{\sigma}\|^{1/2})\;\|\nabla\Delta\sigma\| .$$

An utilization of $abc \le (8\varepsilon^{-3})a^4 + (\varepsilon/2)\,b^2 + (\varepsilon/2)\,c^4$, $\varepsilon > 0$, leads to

$$(2.16) \qquad \frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\|\nabla\Delta\sigma\|^2 \le c\,\varepsilon\|D\bar{v}\|_1^2 + + c_1\,\varepsilon\|\nabla\Delta\sigma\|^2 + \frac{c}{\varepsilon^3}\|\bar{v}\|_1^4\,\|\nabla\bar{\sigma}\|_1^2 + + c\,\varepsilon\,\|\nabla\Delta\bar{\sigma}\|^2 + \frac{c}{\varepsilon^3}\|\bar{v}\|_1^2\,\|\nabla\bar{\sigma}\|_1^4 .$$

Hence for ε verifying (2.13) one has

$$(2.17) \quad \frac{d}{dt}\|\Delta\sigma\|^2 + \lambda\,\|\nabla\Delta\sigma\|^2 \le \frac{c}{\varepsilon^3}(\|\bar{v}\|_1^2\,\|\nabla\bar{\sigma}\|_1^4 + + \|\bar{v}\|_1^4\,\|\nabla\bar{\sigma}\|_1^2) + c_9\varepsilon\,(\|D\bar{v}\|_1^2 + \|\nabla\Delta\bar{\sigma}\|^2),$$

where the constants c depend only on Ω. This proves inequality (2.12). Recall that $\|\sigma\|_2 \le c\|\Delta\sigma\|$, $\|\sigma\|_3 \le c\|\nabla\Delta\sigma\|$.

Remark. One could also consider the linearized equation $D_t\,\sigma + \bar{v}\cdot\nabla\sigma - \lambda\Delta\sigma = 0$ instead of $(2.11)_3$; then estimate (2.17) holds with $\bar{\sigma}$ replaced by σ and without the term $c_9\varepsilon\,\|\nabla\Delta\bar{\sigma}\|^2$. In this case the solution σ of the linearized problem verifies the maximum principle (which doesn't hold for the solution of $(2.11)_3$). However, the linearization $(2.11)_3$ seems to be more in keeping with the linearization $(2.11)_1$. Besides, the maximum principle will be recovered for the solution of the full nonlinear problem $(2.10)_3$.

3. The nonlinear problem. Local existence. We will not take care of the explicit dependence on μ,λ,m,M; some of the constants c, c_k, depend on these fixed quantities. In order to simplify the equations, we denote by K_0, K_1, $K_2,\ldots$, constants depending on the norms of the initial data $\|v_0\|_V$ and $\|\sigma_0\|_2$.

In this section we solve (2.10) by proving the existence of a fixed point $(\rho,v) = (\bar\rho,\bar v)$ for system (2.11). Define

$$K_1 \equiv \{\bar v : \bar v_{|t=0} = v_o(x),\ \|\bar v\|^2_{L^2(0,T;H^2)} +$$

$$+ \|\bar v\|^2_{C(0,T;V)} + \|\bar v'\|^2_{L^2(0,T;H)} \le 2\, C_4\, \|v_o\|^2_V \}$$

$$K_2 \equiv \{\bar\sigma : \bar\sigma_{|t=0} = \sigma_o(x),\ \|\bar\sigma\|^2_{L^2(0,T;H^3_N)} + \|\bar\sigma\|^2_{C(0,T;H^2_N)}$$

$$\le 2\, c_2\, \|\sigma_o\|^2_2,\ \|\sigma'\|_{L^2(0,T;H^1)} \le K_o\ ,$$

$$\|\bar\sigma - \sigma_o\|_{C(\bar Q_T)} \le \frac{m}{2} \}\ ,$$

where $C_4 \equiv \mu[\min\{\mu, M, (m\mu^2)/(4M^2)\}]^{-1}$ and $K_o \equiv \lambda\sqrt{2c_2}\,\|\sigma_o\|_2 + \bar c\,\sqrt{4c_2c_4}\,\|v_o\|_V\,\|\sigma_o\|_2$.
Here $\bar c = \bar c(\Omega)$ is a positive constant such that

(3.1) $$\|\bar v \cdot \bar w\|_1 \le \bar c\, \|\bar v\|_V\, \|\bar w\|_2\ ,\ \forall\, \bar v \in V,\ \bar w \in H^2\ .$$

Note that for every $\bar\sigma \in K_2$ one has in Q_T

(3.2) $$\frac{m}{2} \le \bar\rho(t,x) \le M + \frac{m}{2}\ .$$

We now evaluate the L^2 norm of $\bar F \equiv F(\bar\rho,\bar v)$. By using Sobolev's embeding theorem $H^1 \hookrightarrow L^6$ and Hölder's inequality one easily gets

(3.3) $$\|F(\bar\rho,\bar v)\|^2 \le c\,\|\bar v\|_1^3\,\|D\bar v\|_1 + c\,\|\bar v\|_1^2\,\|D^2\bar\sigma\|\cdot$$
$$\cdot\|D^2\bar\sigma\|_1 + c\,\|\bar v\|_1\,\|D\bar v\|_1\,\|D\bar\sigma\|_1^2 +$$
$$+ c\,\|D\bar\sigma\|_1^3\,\|D^2\bar\sigma\|_1 + c\,\|D\sigma\|_1^6 +$$
$$+ c\,\|f\|^2\ .$$

Consequently,

(3.4) $$\|F(\bar\rho,\bar v)\|^2_{L^2(0,T;H)} \le c\,(\|v_o\|_V + \|\sigma_o\|_2)^{7/2}\,T^{1/2} +$$
$$+ c\,\|\sigma_o\|_2^6\,T + c\,\|f\|^2_{L^2(0,T;H)}\ .$$

Hence, by using (2.3), it follows that the solution v of $(2.11)_1$, $(2.11)_2$ verifies

$$(3.5)\quad \|v\|^2_{C(0,T;V)} + \|v\|^2_{L^2(0,T;H^2)} + \|v'\|^2_{L^2(0,T;L^2)} \leq$$

$$\leq C_4 \|v_o\|^2_V + K_1(\sqrt{T} + T) + C \|f\|^2_{L^2(0,T;H)} .$$

On the other hand, from (2.12),

$$(3.6)\quad \|\sigma\|^2_{C(0,T;H^2_N)} + \|\sigma\|^2_{L^2(0,T;H^3_N)} \leq$$

$$\leq \{ C_2 \|\sigma_o\|^2_2 + K_3 \varepsilon^{-3} T + K_4 \varepsilon \} .$$

Now we fix $\varepsilon > 0$ such that $K_4 \varepsilon \leq 2^{-1} C_2 \|\sigma_o\|^2_2$. Finally, by fixing a sufficiently small $T > 0$, it follows that $v \in K_1$, $\sigma \in K_2$. The estimate for $D_t \sigma$ follows by using equation $(2.11)_3$ and (3.1). The estimate for the sup norm of $\sigma - \sigma_o$ in $\bar{Q}_T$ is proved as follows:

Clearly, $\|\sigma(t) - \sigma_o\|_1 \leq \int_o^t \|\sigma'(s)\|_1\, ds \leq K_o T^{1/2}$. On the other hand $\|\sigma(t) - \sigma_o\|_{C(\bar{\Omega})} \leq C_5 \|\sigma(t)-\sigma_o\|_1^{1/3} \|\sigma(t)-\sigma_o\|_2^{2/3}$, where C_5 depends only on Ω; recall that $H^{5/3}(\Omega) \hookrightarrow C(\bar{\Omega})$. Consequently

$$\|\sigma - \sigma_o\|_{C(\bar{Q}_T)} \leq C_5 K_o^{1/3} T^{1/6} (C_6 \sqrt{2C_2} \|\sigma_o\|_2 + \|\sigma_o\|_2)^{2/3} ,$$

where $C_6 = C_6(\Omega)$ *is a positive constant*, such that $\|\sigma\|_{C(\bar{\Omega})} \leq C_6 \|\sigma\|_2$, $\forall \sigma \in H^2_N(\Omega)$. Hence, by choosing (if necessary) a smaller value for T, one gets $\|\sigma - \sigma_o\|_{C(\bar{Q}_T)} \leq m/2$.

Now we utilize Shauder's fixed point theorem. Clearly $K \equiv K_1 + K_2$ is a convex, compact set in $L^2(0,T;L^2) \times L^2(0,T;L^2)$. Let us denote by Φ the map $\Phi(\bar{\rho},\bar{v}) = (\rho,v)$, defined by (2.11). Since $\Phi(K) \subset K$, it is sufficient to prove that $\Phi : K \to K$ is continuous in the L^2 topology. If $\bar{v}_n \to \bar{v}$ in $L^2(Q_T)$, $\bar{\rho}_n \to \bar{\rho}$ in $L^2(Q_T)$, is follows by compactness arguments that $\bar{v}_n \to \bar{v}$ weakly in $L^2(0,T;H^2)$ and in $H^1(0,T;L^2)$, and that $\nabla\bar{\rho}_n \to \nabla\bar{\rho}$ weakly in $L^2(0,T;H^2)$ and in $H^1(0,T;L^2)$. In particular $\bar{\rho}_n$ is a bounded sequence in $H^{1/2+\varepsilon_1}(0,T;H^{2-\varepsilon_2}) \hookrightarrow C^{o,\alpha}(\bar{Q}_T)$ [3], for suitable positive ε_1, ε_2, α.

[3] α-Hölder continuous functions in $\bar{Q}_T$.

Hence $\bar{\rho}_\nu \to \bar{\rho}$ uniformly in $\bar{Q}_T$. Moreover, $\bar{v}_n$ and $\nabla\bar{\rho}_n$ are bounded in $H^{1/2}(0,T;H^1)$ and in $H^{1/2}(0,T;H^1)$ respectively. Hence $\bar{v}_n \to \bar{v}$ and $\nabla\bar{\rho}_n \to \nabla\bar{\rho}$ strongly in the L^4 topology. It follows from (2.8) that $F(\bar{\rho}_n, \bar{v}_n) \to F(\bar{\rho},\bar{v})$ as a distribution in Q_T; consequently $F(\bar{\rho}_n, \bar{v}_n) \to F(\bar{\rho},\bar{v})$ weakly in $L^2(Q_T)$, because $F(\bar{\rho}_n,\bar{v}_n)$ is a bounded sequence in this space. Analogously, $\bar{v}_n \cdot \nabla\bar{\rho}_n \to \bar{v} \cdot \nabla\bar{\rho}$ strongly in $L^2(Q_T)$. It follows from the linear equations (2.11) that $v_n \to v$ and $\rho_n \to \rho$ in $L^2(Q_T)$ and $L^2(Q_T)$, respectively. Hence Φ is continuous. This finishes the proof of the existence of a local solution. Uniqueness will be proved in Section 5.

4. Global solutions. Asymptotic behavior.

In this section the constants C_K depend at most on Ω and on the quantities μ, λ and $\hat{\rho}$, i.e. on the total amount of mass $|\Omega|\,\hat{\rho}$. We assume that

$$\|\sigma_o\|_2 < (2\bar{C})^{-1}\,\hat{\rho}, \tag{4.1}$$

hence $\hat{\rho}/2 < m < M < 3\,\hat{\rho}/2$. Let (ρ,v) be a solution of (1.1). From (2.6) for $\varepsilon_o = (4M^2)^{-1} m\,\mu$, and from (3.3) one gets

$$\frac{m}{2}\|D_t v\|^2 + \frac{\mu}{2}\frac{d}{dt}\|v\|_V^2 + \frac{m\mu^2}{8\,M^2}\|Av\|^2 < \tag{4.2}$$

$$< C\,(\|v\|_1^2 + \|\sigma\|_2^3)\,(\|v\|_2 + \|\sigma\|_3) + C\,\|\sigma\|_2^6 + C\,\|f\|^2 \,,$$

where C depends only on Ω, μ, $\hat{\rho}$. On the other hand, from (2.17) for $\varepsilon = (2C_9)^{-1}\lambda$, one gets

$$\frac{d}{dt}\|\Delta\sigma\|^2 + \frac{\lambda}{2}\|\nabla\Delta\sigma\|^2 < C\,(\|v\|_1^6 + \|\sigma\|_2^6)\,. \tag{4.3}$$

From (4.2) and (4.3) it easily follows that

$$\frac{d}{dt}\left(\frac{\mu}{2}\|v\|_V^2 + \|\Delta\sigma\|^2\right) + \frac{m}{2}\|D_t v\|^2 +$$

$$+ \frac{m\mu^2}{16M^2}\|Av\|^2 + \frac{\lambda}{4}\|\nabla\Delta\sigma\|^2 < C(\|v\|_V^6 + \|\Delta\sigma\|^6 + \|f\|^2)\,.$$

In particular, since $\|Av\| \geq C\|v\|_V$ and $\|\nabla\Delta\sigma\| \geq C\|\Delta\sigma\|$, one has

$$\frac{d}{dt}\left(\|v\|_V^2 + \|\Delta\sigma\|^2\right) \leq -\left[C_{10}-C_{11}\left(\|v\|_V^2 + \|\Delta\sigma\|^2\right)^2\right]\cdot \tag{4.4}$$
$$\cdot\left(\|v\|_V^2 + \|\Delta\sigma\|^2\right) + C_{12}\ \|f\|^2.$$

Hence $(4.5)_1$ below holds for every $t \in [0,+\infty[$ if

$$\begin{cases} C_{11}\left(\|v_o\|^2 + \|\Delta\sigma_o\|^2\right)^2 < \dfrac{C_{10}}{2}\ \\ C_{12}\ \|f\|^2_{L^\infty(0,+\infty;H)} < \dfrac{C_{10}}{2}\sqrt{\dfrac{C_{10}}{2C_{11}}}\ . \end{cases} \tag{4.5}$$

In fact, if $C_{11}\left(\|v(t)\|_V^2 + \|\Delta\sigma(t)^2\|\right)^2 = C_{10/2}$ it must be, from (4.4), that $\frac{d}{dt}\left(\|v(t)\|_V^2 + \|\Delta\sigma(t)\|^2\right) < 0$.

Let us now prove the last assertion in theorem A. Under the hypothesis $(4.5)_1$, it follows from (4.4) that

$$\frac{d}{dt}\left(\|v\|_V^2 + \|\Delta\sigma\|^2\right) \leq -\frac{C_{10}}{2}\left(\|v\|_V^2 + \|\Delta\sigma\|^2\right)\ .$$

this proves (1.6) . □

5. Uniqueness. We prove that the solution (ρ,v) of problem (1.1) is unique in the class in which existence was proved; see Theorem A. We remark that more careful calculations lead to uniqueness in a larger class.

Let (ρ,v), $(\bar\rho,\bar v)$ be two solutions of problem (1.1), (1.2) and put $u = v-\bar v$, $\eta = \rho-\bar\rho$. By subtracting the equations $(2.10)_1$ written for (ρ,v) and $(\bar\rho,\bar v)$ respectively, and by taking the inner product with u in H one gets

$$\frac{1}{2}\frac{d}{dt}(\rho u,u) + \mu\ \|u\|_V^2 = -\frac{1}{2}(v\cdot\nabla\rho,u^2) +$$
$$+\frac{\lambda}{2}(\Delta\rho,u^2) - (\mu,D_t\bar v\cdot u) + (F-\bar F,u)\ .$$

By using $(\Delta\rho,u^2) = -(\nabla\rho,Du^2)$, we show that

$$\frac{1}{2}\frac{d}{dt}(\rho u,u) + \frac{\mu}{2}\|u\|_V^2 \leq \frac{1}{2}\|v\|_\infty\|\nabla\rho\|_\infty\|u\|^2 + \tag{5.1}$$
$$+\frac{\lambda^2}{\mu}\|\nabla\rho\|_\infty^2\ \|u\|^2 + C\|D_t\bar v\|^2\ \|u\|^2 +$$
$$+\frac{\lambda}{4}\|\Delta\eta\|^2 + (F-\bar F,u)\ .$$

On the other hand, by subtracting equations $(2.10)_3$ written for (ρ,v) and for $(\bar{\rho},\bar{v})$ respectively, and by taking the inner product with $\Delta\eta$ in $L^2(\Omega)$ one gets

$$(5.2)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla\eta\|^2 + \frac{\lambda}{2}\|\Delta\eta\|^2 \leq c\,\|\nabla\bar{\rho}\|_\infty^2\,\|u\|^2 + \\ + c\,\|v\|_\infty^2\,\|\nabla\eta\|^2 .$$

By adding (5.1) and (5.2) it follows that

$$(5.3)\qquad \frac{d}{dt}[(\rho u,u) + \|\nabla\eta\|^2] + \mu\,\|u\|_V^2 + \frac{\lambda}{2}\|\Delta\eta\|^2 \leq \\ \leq \theta_1(t)\left(\|u\|^2 + \|\nabla\eta\|^2\right) + (F-\bar{F},u),$$

where $\theta_1(t)$ is a real integrable function on $[0,T]$.

On the other hand, by using Sobolev's embeeding theorems and Hölder's inequality (and also $ab \leq \varepsilon a^2+\varepsilon^{-1}b^2$) the reader easily verifies that given $\varepsilon > 0$ there exists an integrable real function $\theta_2(t)$ (dependent on ρ, $\bar{\rho}$, v, $\bar{v}$ and on ε) such that

$$(5.4)\qquad |(F-\bar{F},u)| \leq \theta_2(t)\,\|u\|^2 + \varepsilon\,(\|\eta\|_2^2+\|u\|_1^2)\ .$$

By using $\|u\|^2 \leq m^{-1}(\rho u,u)$, (5.3) and (5.4) it follows that

$$\frac{d}{dt}[(\rho u,u) + \|\nabla\eta\|^2] \leq (\theta_1(t) + \theta_2(t))\,[(\rho u,u) + \|\nabla\eta\|^2)].$$

Uniqueness follows now from Gronwall's lemma and from $u_{|t=0} = 0$, $\eta_{|t=0} = 0$. $\square$

REFERENCES

1. H. Beirão-da-Veiga, R. Serapioni, A. Valli "On the Motion of Non-homogeneous fluids in the Presence of Diffusion", J. Math. Anal. Appl., 85 (1982), 179-191.

2. D.A. Frank-Kamenetskii "Diffusion and Heat Transfer in chemical Kinetics", Plenum ed., New York, London (1969).

3. E. Gagliardo "Caretterizzazioni delle tracce sulla Frontiera Relative ad alcune classi di Functioni in N Variabili", Rend. Sem. Mat. Univ. Padova, 27 (1957), 284-305.

4. V.N. Ignat'ev, B.G. Kuznetsov "A Model for the Diffusion of a Turbulent Boundary Layer in a Polymer", Čisl. Metody Meh. Splošn. Sredy, 4 (1973), 78-87 (in russian).

5. A.V. Kazhikhov, Sh. Smagulov, "The Correctness of Boundary-Value Problems in a Diffusion Model of an Inhomogeneous Liquid", Sov. Phys. Dokl. , 22 (1977), 249-250.

6. A.V. Kazhikhov, Sh. Smagulov, "The Correctness of Boundary-Value Problems in a Certain Diffusion Model of an Inhomogeneouse Fluid", Čisl. Metody Meh. Splošn. Sredy, 7 (1976), 75-92.

7. O.A. Ladyzhenskaya, "The Mathematical Theory of Viscous Incompressible Flow", Gordon and Breach, New York (1969).

8. J.L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires", Dunod, Gauthier-Villars, Paris (1969).

9. J.L. Lions, E. Magenes "Problèmes aux limites nonhomogènes et applications", Vol. I,II, Dunod, Paris (1968).

10. G. Prouse, A. Zaretti "On the Inequalities Associated to a Model of Graffi for the Motion of a Mixture of two viscous Incompressible Fluids", preprint Departimento di Matematica - Politecuico di Milano (1982).

11. P. Secchi "On the Initial Value Problem for the Equations of Motion of Viscous Incompressible Fluids in Presence of Diffusion", Boll. U.M.I., 1-B(1982) 1117-1130.

12. R. Temam "Navier-Stokes Equations", North Holland Amsterdam (1977).

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We consider the motion of a mixture of two fluids, with a diffusion effect obeying Fick's law; for the derivation of the model (1.1) see Section 1 and references [2], [4],[5] and [6]. We consider the full non-linear problem (i.e., we don't omit the term λ^2 term in equation (1.1_1). Moreover, we don't assume that λ/μ is small. We prove the existence of a (unique) local

ABSTRACT (Continued)

solution, the existence of a global solution for small data, and the exponential decay to the equilibrium solution; see Theorem A, Section 1.